

# Phase transitions of the BTZ black hole in new massive gravity

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## Abstract

We investigate thermodynamics of the BTZ black hole in new massive gravity explicitly. For  $m^2\ell^2 > 1/2$  with  $m^2$  the mass parameter of fourth-order terms and  $\ell^2$  AdS<sub>3</sub> curvature radius, the Hawking-Page phase transition occurs between the BTZ black hole and AdS (thermal) soliton. For  $m^2\ell^2 < 1/2$ , however, this transition unlikely occurs but a phase transition between the BTZ black hole and the massless BTZ black hole is possible to occur. We may call the latter as the inverse Hawking-Page phase transition and this transition is favored in the new massive gravity.

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# 1 Introduction

A black hole could be rendered thermodynamically stable by placing it in four-dimensional anti-de Sitter ( $\text{AdS}_4$ ) spacetimes because  $\text{AdS}_4$  spacetimes play the role of a confining box. Then, it is a natural question to ask how a stable black hole with positive heat capacity could emerge from thermal radiation through a phase transition. This was known to be the Hawking-Page (HP) phase transition between thermal radiation (TR) and Schwarzschild- $\text{AdS}_4$  black hole (SAdS) [1, 2]. It has shown one typical example of the first-order phase transition ( $\text{TAdS} \rightarrow \text{small SAdS} \rightarrow \text{large SAdS}$ ) in the gravitational system. In the last two decades, its higher dimensional extension and its holographic dual to confinement-deconfinement transition were the hottest issues [3].

In order to study the HP phase transition in Einstein gravity, we need to know the Arnowitt-Deser-Misner (ADM) mass [4], the Hawking temperature, and the Bekenstein-Hawking (BH) entropy. These are combined to give the on-shell free energy in canonical ensemble which determines the global thermodynamic stability. The other important quantity is the heat capacity which determines the local thermodynamic stability. Employing the Euclidean action formalism, one easily finds these quantities [5]. However, a complete computation of the thermodynamic quantities was limited in fourth-order gravity because one has encountered some difficulty to compute their conserved quantities in asymptotically AdS spacetimes.

In three dimensions, either third-order gravity (topologically massive gravity [6]) or the fourth-order gravity (new massive gravity [7]) is essential to describe a spin-2 graviton because the Einstein gravity is a gauge theory without propagating degrees of freedom. Recently, there was a significant progress on computation of mass and thermodynamic quantities by using the Abbot-Deser-Tekin (ADT) method [8, 9, 10]. One has to recognize that all ADT thermodynamic quantities except the Hawking temperature depend on a mass parameter  $m^2$ . Hence, for  $m^2\ell^2 > 1/2$ , all thermodynamic properties are dominantly determined by Einstein gravity, while for  $m^2\ell^2 < 1/2$ , all thermodynamic properties are dominantly determined by purely fourth-order curvature term. More recently, it was shown that the HP phase transition (thermal soliton  $\rightarrow$  BTZ black hole) occurs for  $m^2\ell^2 > 1/2$  in new massive gravity by computing off-shell free energies of black hole and soliton [11]. However, the role of the massless BTZ black hole was missed. The former can be completely understood if the massless BTZ black hole is introduced as a mediator. Furthermore, the present is a turnaround time to explore the  $m^2\ell^2 < 1$  case of black hole thermodynamics if one wishes to study the black hole thermodynamics by employing the massive gravity theory.

On the other hand, we would like to mention that the stability condition of the BTZ

black hole in the new massive gravity turned out to be  $m^2\ell^2 > 1/2$  regardless of the horizon size  $r_+$ , while the instability condition is given by  $m^2\ell^2 < 1/2$  [12]. For  $m^2\ell^2 < 1/2$ , the BTZ black hole is thermodynamically unstable because of  $C_{\text{ADT}} < 0$  and  $F_{\text{ADT}}^{\text{on}} > 0$  as well as it is classically unstable against the metric perturbations. The latter indicates a perturbative instability of the BTZ black hole arisen from the massiveness of graviton. It implies a deep connection between thermodynamic instability and classical instability for the BTZ black hole only for the new massive gravity [13]. Also, it suggests that the phase transition for  $m^2\ell < 1/2$  is quite different from that of the  $m^2\ell > 1/2$  case. Here, we wish to explore the presumed phase transition and it will be compared with the Hawking-Page phase transition for the  $m^2\ell > 1/2$  case.

## 2 Thermodynamics of the BTZ black hole

We introduce the new massive gravity (NMG) composed of the Einstein-Hilbert action with a cosmological constant  $\lambda$  and fourth-order curvature terms [7]

$$S_{\text{NMG}} = S_{\text{EH}} + S_{\text{FOT}}, \quad (1)$$

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} (R - 2\lambda) \quad (2)$$

$$S_{\text{FOT}} = -\frac{1}{16\pi G m^2} \int d^3x \sqrt{-g} \left( R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right), \quad (3)$$

where  $G$  is a three-dimensional Newton constant and  $m^2$  a mass parameter with mass dimension 2. In the limit of  $m^2 \rightarrow \infty$ ,  $S_{\text{NMG}}$  recovers the Einstein gravity, while  $S_{\text{NMG}}$  reduces to purely fourth-order gravity in the limit of  $m^2 \rightarrow 0$ . The Einstein equation is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0, \quad (4)$$

where

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu R - \frac{\Box R}{2} g_{\mu\nu} + 4R_{\mu\rho\nu\sigma} R^{\rho\sigma} - \frac{3R}{2} R_{\mu\nu} - R_{\rho\sigma}^2 g_{\mu\nu} + \frac{3R^2}{8} g_{\mu\nu}. \quad (5)$$

The BTZ black hole solution to Eq.(4) is given by [14, 15]

$$ds_{\text{BTZ}}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \quad f(r) = -M + \frac{r^2}{\ell^2} \quad (6)$$

when satisfying a condition of  $1/\ell^2 + \lambda + 1/(4m^2\ell^4) = 0$  with  $\ell^2$  the curvature radius of  $\text{AdS}_3$  spacetimes. Here  $M$  is related to the ADM mass of black hole. The horizon radius  $r_+$  is determined by the condition of  $f(r_+) = 0$ .

On the other hand, the linearized equation to (4) upon choosing the transverse-traceless gauge of  $\bar{\nabla}^\mu h_{\mu\nu} = 0$  and  $h^\mu{}_\mu = 0$  leads to the fourth-order linearized equation for the metric perturbation  $h_{\mu\nu}$

$$\left(\bar{\nabla}^2 - 2\Lambda\right)\left[\bar{\nabla}^2 - 2\Lambda - \mathcal{M}^2(m^2)\right]h_{\mu\nu} = 0, \quad \Lambda = -\frac{1}{\ell^2} \quad (7)$$

which might imply the two second-order linearized equations

$$\left(\bar{\nabla}^2 - 2\Lambda\right)h_{\mu\nu} = 0, \quad (8)$$

$$\left[\bar{\nabla}^2 - 2\Lambda - \mathcal{M}^2(m^2)\right]h_{\mu\nu} = 0, \quad (9)$$

where the mass squared  $\mathcal{M}^2$  of a massive spin-2 graviton is given by

$$\mathcal{M}^2(m^2) = m^2 - \frac{1}{2\ell^2} \rightarrow \frac{\mathcal{M}^2}{m^2} = 1 - \frac{1}{2m^2\ell^2}. \quad (10)$$

Eq.(9) describes a massive graviton with  $2(6 - 4 = 2)$  DOF propagating around the BTZ black hole under the gauge, while Eq.(8) indicates a non-propagating spin-2 graviton in the Einstein gravity. This explains clearly why the NMG describes a massive graviton with 2 DOF. The presence of  $S_{\text{FOT}}$  distinguishes the NMG from the Einstein gravity because it generates 2 DOF. At this stage, we briefly mention the stability of the BTZ black hole in the NMG. The stability condition of the BTZ black hole in the NMG turned out to be  $m^2\ell^2 > 1/2 (\mathcal{M}^2 > 0)$  regardless of the horizon size  $r_+$ , while the instability condition is given by  $m^2\ell^2 < 1/2 (\mathcal{M}^2 < 0)$  [12]. This is valid for the NMG, not for the Einstein gravity.

Now we derive all thermodynamic quantities. The Hawking temperature is found to be

$$T_{\text{H}} = \frac{f'(r_+)}{4\pi} = \frac{r_+}{2\pi\ell^2} \quad (11)$$

which is the same for the Einstein gravity. Using the ADT method, one could derive the mass [16], heat capacity, entropy [17], and on-shell free energy

$$M_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2}M, \quad C_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2}C, \quad S_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2}S_{\text{BH}}, \quad F_{\text{ADT}}^{\text{on}} = \frac{\mathcal{M}^2}{m^2}F^{\text{on}} \quad (12)$$

For  $G = 1/8$ , thermodynamic quantities in Einstein gravity are given by [18, 19, 20]

$$M = \frac{r_+^2}{\ell^2}, \quad C = 4\pi r_+, \quad S_{\text{BH}} = 4\pi r_+, \quad F^{\text{on}} = M - T_{\text{H}}S_{\text{BH}} = -\frac{r_+^2}{\ell^2} = -M \quad (13)$$

which are positive regardless of the horizon size  $r_+$  except that the free energy is negative. This means that the BTZ black hole is thermodynamically stable in Einstein gravity. Here we check that the first-law of thermodynamics is satisfied as

$$dM_{\text{ADT}} = T_{\text{H}}dS_{\text{ADT}}, \quad (14)$$

as in Einstein gravity

$$dM = T_H dS_{\text{BH}} \quad (15)$$

where ‘ $d$ ’ denotes the differentiation with respect to the horizon size  $r_+$  only. Importantly, we note that in the limit of  $m^2 \rightarrow \infty$  we recover thermodynamics of the BTZ black hole in Einstein gravity, while in the limit of  $m^2 \rightarrow 0$  we recover the black hole thermodynamics in purely fourth-order gravity. The latter is similar to recovering the third-order terms of conformal Chern-Simons gravity from the topologically massive gravity [21] and conformal gravity from the Einstein-Weyl gravity [22, 13].

It is well known that the local thermodynamic stability is determined by the positive heat capacity ( $C_{\text{ADT}} > 0$ ) and the global stability is determined by the negative free energy ( $F_{\text{ADT}}^{\text{on}} < 0$ ). Therefore, we propose that the thermodynamic stability is determined by the sign of the heat capacity while the phase transition is mainly determined by the sign of the free energy.

To investigate a phase transition, we introduce the thermal soliton (TSOL) whose thermodynamic quantities are given by [11]

$$M_{\text{ADT}}^{\text{TSOL}}(m^2) = \frac{\mathcal{M}^2}{m^2} M^{\text{TSOL}}, \quad F_{\text{ADT}}^{\text{TSOL}}(m^2) = \frac{\mathcal{M}^2}{m^2} F^{\text{TSOL}}, \quad (16)$$

where

$$M^{\text{TSOL}} = 1, \quad F^{\text{TSOL}} = -1 \quad (17)$$

with  $G = 1/8$ . We note that a factor of  $\mathcal{M}^2/m^2$  was missed in Ref.[13]. The TSOL corresponds to the spacetime picture of the NS-NS vacuum state [23].

Furthermore, we need the massless BTZ black hole (MBTZ) whose thermodynamic quantities all are zero as [19, 24]

$$M_{\text{ADT}}^{\text{MBTZ}} = 0, \quad F_{\text{ADT}}^{\text{MBTZ}} = 0. \quad (18)$$

The MBTZ is called the spacetime picture of the R-R vacuum state.

In addition to a global mass parameter  $m^2$ , we introduce five parameters to describe the phase transition in NMG. These are included as

- $M$  : order parameter,
- $T_H(M)$  : order parameter (onshell temperature),
- $T$  : control parameter (offshell temperature),
- $F_{\text{ADT}}^{\text{on}}(m^2, M)$  : increasing (decreasing) black hole via equilibrium process,
- $F_{\text{ADT}}^{\text{off}}(m^2, M, T)$  : increasing (decreasing) black hole via nonequilibrium process,

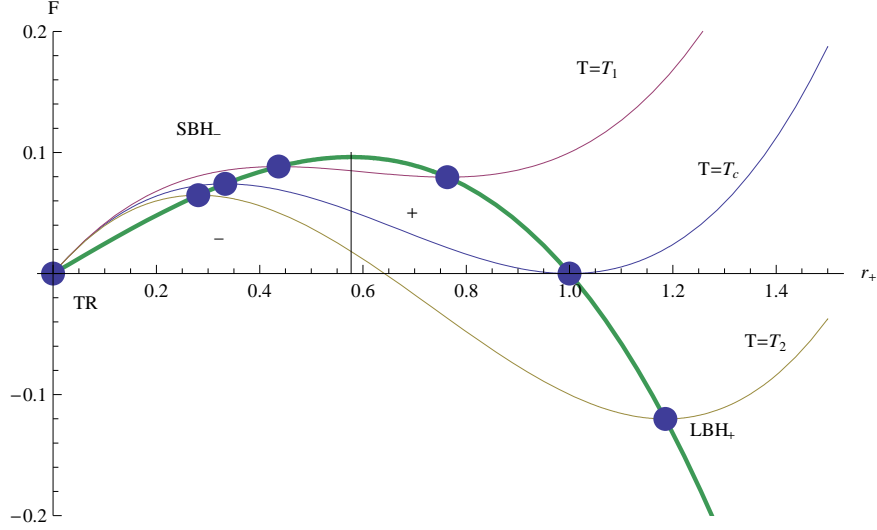


Figure 1: Hawking-Page phase transition for the SAdS with  $l = 1$ : the thick curve represents the on-shell free energy  $F_{\text{SAdS}}^{\text{on}}(r_+)$ , while three thin curves denote the off-shell free energy  $F_{\text{SAdS}}^{\text{off}}(r_+, T)$  with three temperatures  $T = T_1 = 0.9\pi^{-1}(< T_c)$ ,  $T_c = \pi^{-1}$ ,  $T_2 = 1.1\pi^{-1}( > T_c)$ .  $- (+)$  denote negative (positive) heat capacity. The  $\text{SBH}_-$  is bounded in the right by the line at  $r_+ = r_* = 0.57$ .

where off-shell (on-shell) mean equilibrium (non-equilibrium) configurations. In general, the equilibrium process implies a reversible process, while the non-equilibrium process implies a irreversible process. The off-shell free energy corresponds to a generalized free energy which is similar to a temperature-dependent scalar potential  $V(\varphi, T)$  for a simple model of thermal phase transition where  $\varphi$  is the order parameter and  $T$  is a control parameter.

### 3 Phase transitions

#### 3.1 HP phase transition

Before we proceed, we understand intuitively how the original HP transition occurs between Schwarzschild-AdS<sub>4</sub> black hole (SAdS) and thermal radiation (TR). For  $G_4 = 1$ , the ADM mass, Hawking temperature, and the Bekenstein-Hawking entropy are given by

$$M_{\text{SAdS}}(r_+) = \frac{1}{2} \left( r_+ + \frac{r_+^3}{l^2} \right), \quad T_{\text{H}}(r_+) = \frac{1}{4\pi} \left( \frac{1}{r_+} + \frac{3r_+}{l^2} \right), \quad S_{\text{BH}} = \pi r_+^2. \quad (19)$$

In addition, the heat capacity and on-shell free energy are given by

$$C_{\text{SAdS}}(r_+) = 2\pi r_+^2 \left( \frac{3r_+^2 + l^2}{3r_+^2 - l^2} \right), \quad F_{\text{SAdS}}^{\text{on}}(r_+) = \frac{r_+}{4} \left( 1 - \frac{r_+^2}{l^2} \right), \quad (20)$$

where  $C_{\text{SAdS}}$  blows up at  $r_+ = r_* = l/\sqrt{3}$  (heat capacity is changed from  $-\infty$  to  $\infty$  at  $r_+ = r_*$ ). The critical temperature  $T_c = T_H(r_+)|_{r_+=r_c} = \frac{1}{\pi l}$  is determined from the condition of  $F_{\text{SAdS}}^{\text{on}}(r_+) = 0$  for  $r_+ = r_c = l$ . The TR is located at  $r_+ = 0$  in this picture.

In studying thermodynamic stability, two relevant quantities are the heat capacity  $C_{\text{SAdS}}$  which determines thermally local stability (instability) for  $C_{\text{SAdS}} > 0$  ( $C_{\text{SAdS}} < 0$ ) and on-shell free energy  $F_{\text{SAdS}}^{\text{on}}$  which determines the thermally global stability (instability) for  $F_{\text{SAdS}}^{\text{on}} < 0$  ( $F_{\text{SAdS}}^{\text{on}} > 0$ ). A SAdS is thermodynamically stable only if  $C_{\text{SAdS}} > 0$  and  $F_{\text{SAdS}}^{\text{on}} < 0$ . For simplicity, we choose  $l = 1$ . We observe that the on-shell free energy (thick curve in Fig. 1) is maximum at  $r_+ = r_* = 0.57$  and zero at  $r_+ = r_c = 1$  which determines the critical temperature. For  $r_+ > r_c$ , one finds negative free energy.

In order to investigate the HP phase transition, one has to introduce the off-shell free energy as a function of  $r_+$  and  $T$

$$F_{\text{SAdS}}^{\text{off}}(r_+, T) = M_{\text{SAdS}}(r_+) - T S_{\text{BH}}(r_+), \quad (21)$$

where  $T$  plays a role of control parameter for taking a phase transition.

For  $T = T_2 > T_c$ , the process of phase transition is shown in Fig. 1 explicitly. In this case, one starts with TR ( $\bullet$ ) at  $r_+ = 0$  in AdS space and a small black hole ( $\bullet$ :SBH $_-$ ) appears at  $r_+ = 0.28$ . Here the SBH $_-$  denotes a unstable small black hole with  $C_{\text{SAdS}} < 0$  and  $F_{\text{SAdS}}^{\text{on}} > 0$ . This plays a role of the mediator. Then, since the heat capacity changes from  $-\infty$  to  $\infty$  at  $r_+ = r_*$ , the large black hole ( $\bullet$ :LBH $_+$ ) finally comes out as a stable object at  $r_+ = 1.19$ . Here the LBH $_+$  represents a globally stable black hole because of  $C_{\text{SAdS}} > 0$  and  $F_{\text{SAdS}}^{\text{on}} < 0$ . Actually, there is a change of the dominance at the critical temperature  $T = T_c$ : from TR to SAdS [1]. This is called the HP phase transition as a typical example of the first-order transition in the gravitational system: TR  $\rightarrow$  SBH $_- \rightarrow$  LBH $_+$ .

For  $T = T_1 (< T_c)$ , the free energy  $F_{\text{SAdS}}^{\text{on}}(0) = 0$  of TR is the lowest state, while for the  $T = T_2 (> T_c)$  case, the lowest state is the free energy  $F_{\text{SAdS}}^{\text{on}}(1.19) < 0$  for the large black hole. Hence, for  $T_1 < T_c$ , the ground state is TR, whereas for  $T_2 > T_c$ , the ground state is LBH $_+$ . There is no phase transition for  $T = T_1$ .

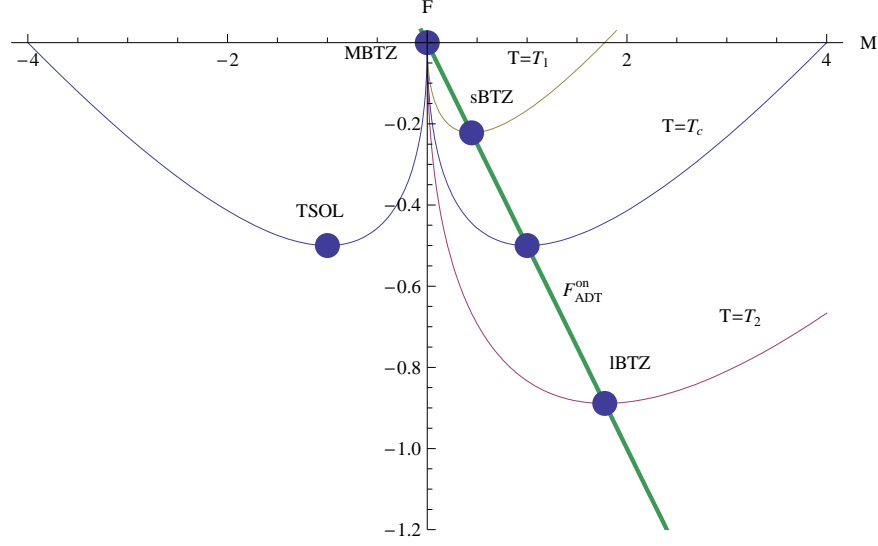


Figure 2: On-shell free energy  $F_{\text{ADT}}^{\text{on}}(m^2 = 1, M) = -M/2$  for the BTZ and  $F_{\text{ADT}}^{\text{TSOL}}(m^2 = 1) = -0.5$  for TSOL with  $G_3 = 1/8$  and  $\ell = 1$ . Three off-shell free energies for the BTZ are given by  $F_{\text{ADT}}^{\text{off}}(m^2 = 1, M, T)$  for  $T = T_1 = \frac{1}{3\pi}, T_c = \frac{1}{2\pi}, T_2 = \frac{2}{3\pi}$ . On the other hand, the off-shell free energy for TSOL is  $F_{\text{TSOL}}^{\text{off}}$ . They are connected at the point of  $M = 0$ . Here, the MBTZ located at  $M = 0$  plays a role of the mediator like SBH<sub>-</sub>.

### 3.2 HP transition in NMG

The off-shell free energy for the BTZ black hole is given by

$$F_{\text{ADT}}^{\text{off}}(m^2, M, T) = \frac{\mathcal{M}^2}{m^2} \left( M - 4\pi\ell T\sqrt{M} \right). \quad (22)$$

On the other hand, the off-shell free energy for TSOL takes the form [20, 11]

$$F_{\text{TSOL}}^{\text{off}}(m^2, M) = \frac{\mathcal{M}^2}{m^2} \left( -M - 2\sqrt{-M} \right). \quad (23)$$

It is worth noting that the author has claimed that the phase transition from TSOL to BTZ is possible to occur without introducing (23) [19]. However, the construction of (23) is necessary to show how the transition from TSOL to BTZ occurs nicely [20].

Here we would like to mention that for  $\mathcal{M}^2 > 0$ , the BTZ under consideration are thermally stable because the heat capacity is always positive. Hence, the free energy plays a key role in studying a phase transition between two gravitational configurations. We emphasize that two on-shell free energies  $F_{\text{ADT}}^{\text{on}}$  and  $F^{\text{TSOL}} = -1/2$  are disconnected to each other. To discuss the phase transition with  $\mathcal{M}^2 > 0 (m^2 = 1 > 1/2)$ , we would be better to examine



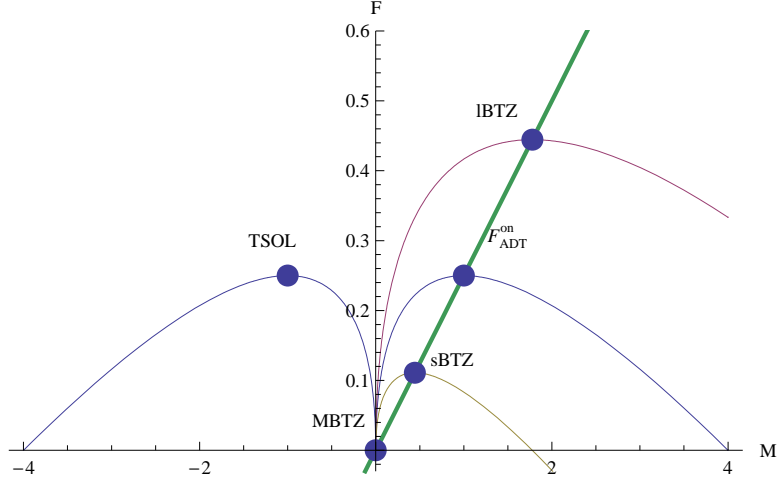


Figure 3: On-shell free energy  $F_{\text{ADT}}^{\text{on}}(m^2 = 0.4, M) = 0.25M$  for the BTZ and  $F_{\text{ADT}}^{\text{TSOL}}(m^2 = 0.4) = 0.25$  for TSOL. Three off-shell free energies for the BTZ are given by  $F_{\text{ADT}}^{\text{off}}(m^2 = 0.4, M, T)$  for  $T = T_1 = \frac{1}{3\pi}, T_c = \frac{1}{2\pi}, T_2 = \frac{2}{3\pi}$ . On the other hand, the off-shell free energy for TSOL is  $F_{\text{TSOL}}^{\text{off}}(m^2 = 0.4, M)$ . Here we observe that the MBTZ located at  $M = 0$  is the ground state.

two off-shell free energies (22) and (23). One finds from Fig. 2 that for  $T = T_1$ , the TSOL (●) is more favorable than the small BTZ (●), while for  $T = T_2$ , the large BTZ (●) is more favorable than TSOL (●). This observation suggests a phase transition (TSOL→MBTZ→BTZ) for  $T > T_c$ . Two off-shell free energies are connected at the point  $M = 0$ . At  $T = T_c$ , the transition from TSOL to BTZ black hole through MBTZ is possible to occur. For  $T = T_1 < T_c$ , the TSOL dominates because of  $F^{\text{TSOL}} < F_{\text{ADT}}^{\text{on}}$ , while for  $T = T_2 > T_c$ , the BTZ dominates because of  $F^{\text{TSOL}} > F_{\text{ADT}}^{\text{on}}$ . This indicates that a change of dominance occurs at the critical temperature  $T = T_c$  [19]. Importantly, this transition could be regarded really as a HP transition because the MBTZ plays a role of the mediator like SBH<sub>-</sub> in the previous HP phase transition [13]. A difference is that the MBTZ is a single extremal state (see Fig.2), whereas SBH<sub>-</sub> has three states depending on  $T$  (see Fig.1). A similarity is that they all are the highest states.

### 3.3 Inverse HP transition in NMG

Generally, increasing black hole ( $\searrow$ ) is induced by absorbing radiations in the heat reservoir, while decreasing black hole ( $\swarrow$ ) is done by Hawking radiations as evaporation process. The former describes the HP phase transition, whereas the latter denotes the inverse Hawking-Page phase transition (IHP).

For  $\mathcal{M}^2 < 0$  case, fourth-order curvature terms (3) contribute dominantly to black hole thermodynamics. In this case, the heat capacity of the BTZ is negative and the on-shell free energy is positive, which means that the BTZ is thermodynamically unstable. This is consistent with the classical instability of the BTZ for  $m^2 < 1/2$  [12]. The other quantities of ADT mass and entropy are negative, which may raise a problem in obtaining a consistent black hole thermodynamics. To avoid this problem, the author in [11] has required  $\mathcal{M}^2 > 0$ . However, at this moment, we could not understand why this problem arises in the ADT approach to the black hole thermodynamics in the NMG. Instead, we would be better to make a progress on the phase transition because this is a main feature of the BTZ in the NMG. Here, we wish to know whether a phase transition from the BTZ to TSOL is possible to occur in NMG for  $\mathcal{M}^2 < 0$  ( $m^2 = 0.4 < 1/2$ ). We find from Fig. 3 that the MBTZ located at  $M = 0$  is always more favorable than BTZ and TSOL because of  $F_{\text{MBTZ}} < F_{\text{ADT}}^{\text{on}}(m^2 = 0.4, M), F^{\text{TSOL}}(m^2 = 0.4)$ . In other words, it corresponds to the ground state. In this case, we might not define a possible phase transition between TSOL and BTZ because the ground state is given by the MBTZ. A possible transition is the IHP (BTZ $\rightarrow$ MBTZ) [19], whereas the TSOL is located at the left branch. This may be possible because the MBTZ is considered as an extremal black hole without size.

## 4 Discussions

We have investigated thermodynamics of the BTZ black hole in new massive gravity completely. We have confirmed that for  $m^2\ell^2 > 1/2$ , the Hawking-Page phase transition occurs between the BTZ black hole and thermal soliton by introducing the massless BTZ black hole. Here the massless BTZ black hole plays a role of the mediator like SBH $_-$  in the original Hawking-Page phase transition and it is the highest state.

On the other hand, for  $m^2\ell^2 < 1/2$ , this transition unlikely occurs but a phase transition between the BTZ black hole and the massless BTZ black hole is possible to occur. In this case, the massless BTZ black hole is the ground state. We call the latter as the inverse Hawking-Page phase transition and this transition is favored in the new massive gravity. This completes phase transitions of the BTZ black hole in the new massive gravity.

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